Multirate technique for explicit Discontinuous Galerkin computations of time domain Maxwell equations on complex geometries

A. Kameni[1], B. Seny[2], L. Pichon[1]

[1] Group of Electrical Engineering of Paris, UMR 8507 CNRS, CentraleSupelec, Université Paris Sud, Université Pierre et Marie Curie, Plateau du Moulon, 91192 Gif-Sur-Yvette Cedex, France

[2] Institute of Mechanics, Materials and Civil Engineering, Université Catholique de Louvain-la-Neuve, 4-6 avenue Georges Lemaître B-1348, Louvain-la-Neuve, Belgique

This paper presents a multirate technique to improve and optimize the time step of a second order $2-$stages Explicit Runge-Kutta scheme (ERK). In this technique, the mesh elements are stored in different groups according to their stable time steps. These groups are sorted into two classes. The bulk groups where the $2-$stages ERK method is applied once or repeatedly. The buffer groups that served to accommodate transition between two bulks groups. This technique is proposed for accelerating explicit discontinuous Galerkin computations of time domain Maxwell equations. An application example on human skull is proposed to show efficiency of this technique to simulate wave propagation on complex geometries.

***Index Terms*—Discontinuous Galerkin method, Explicit Runge-Kutta (ERK) schemes, Maxwell equations, Multirate technique**

I. Introduction

THE development of electromagnetic applications and devices need implementation of efficient tools to solve Maxwell equations in complex geometries. Explicit methods such as the Runge-Kutta (ERK) schemes (ERK) are commonly used to perform electromagnetic modeling of these systems. The time step to ensure convergence depends on the smallest mesh elements and is defined by the CFL (Courant-Friedrichs-Lewy) condition. The presence of small size details leads on fine meshes with different refinement level that drastically increase the computational costs.

The discontinuous Galerkin methods are well adapted for using the local time stepping strategy to reduce the expensive computations by adapting the time step under the local stability conditions required by the classical schemes [1] [2]. The multirate methods are a subset of the local time-stepping schemes family. They allow using of different time steps that are integer ratios of each other to solve a discrete system [3]. This technique has been used with a multi-step Adams-Bashforth scheme to improve Discontinuous Galerkin computations of Maxwell equations [4]. Its implementation using ERK schemes has been recently proposed to accelerate geophysical flows computations [5].

In this paper a multirate technique based on $2-$stages ERK method is presented and applied on Maxwell equations. This approach consist in gathered the mesh elements in different groups that satisfy the CFL condition for a certain range of time steps. These groups are sorted into two classes. The bulk groups where a $2-$stages ERK method is applied once or repeatedly. The buffer groups that served to accommodate transition between two bulks groups and where a $2-$stages ERK scheme is adapted to coincide with the ERK schemes used in the bulks groups. Efficiency of this approach is shown through computations of electric field on human skull.

II. Maxwell equations and Discrete system

Let E, H and J respectively, the electric field and the magnetic field and the current density. They satisfy the Maxwell equations given by (1):

$$\begin{cases} \epsilon\partial_t E - \nabla\times H = -J \\ \mu\partial_t H + \nabla\times E = 0 \end{cases} \tag{1}$$

where ϵ and μ are respectively the permittivity and the permeability of the medium. In a conductive medium, $J=\sigma E$, with σ the conductivity. A nodal Discontinuous Galerkin method based on upwing flux terms is used for the spatial discretization [6]. This leads on an ordinary differential equation on each mesh element T of the domain $\Omega = \cup T$:

$$\begin{cases} \dfrac{du_i}{dt} = f(u_i(t),t) \\ \qquad u_i(0) = u_i^0 \end{cases} \tag{2}$$

where $u_i = (E_i, H_i)$.The explicit Runge-Kutta methods are set to integrate the solution in time.

III. Multirate strategy

A. Construction of the multirate groups

Let's note δt_m and δt_M respectively the time steps according to CFL of the smallest and the biggest elements. In this paper, the multirate groups are built for having their stable time steps of ratio $\kappa = 2$. If the time step of the multirate strategy is $\delta t \in [\delta t_m, \delta t_M]$, the number of multirate groups $N_g = z_g+1$ where z_g is an integer defined by $z_g = log_2\left(\dfrac{\delta t}{\delta t_m}\right)$. A multirate group Ω_z is constituated by elements whose time steps are in $\left[\dfrac{\delta t}{2^{z+1}}, \dfrac{\delta t}{2^z}\right]$, with $z \le z_g$. The time step δt has to be chosen in a range defined as : $\delta t = \max\limits_z(\alpha 2^z \delta t_m)$ such as $\alpha 2^z\delta t_m \le \delta t_M$ with $1/2 \le \alpha \le 1$.

To define the bulk and buffer groups, a tag θ is attributed to each multirate group. This integer is defined by $\theta = 3(z_g - z) + \sigma$, with $\sigma \in \{0, 2\}$, such as $\sigma = 0$ corresponds to the bulk groups and $\sigma = 2$ to the buffer groups. The example in Fig.1 is an illustration that shows two bulk groups of stable time steps Δt_m and $2\Delta t_m$, separated by a buffer group.

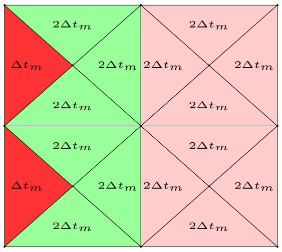

Fig. 1. Example of a mutirate partitioning: a buffer group (green) separates two bulk groups of different stable time steps of ratio 2, group Δt_m (red) and group $2\Delta t_m$ (pink).

B. *Multirate time integration*

The multirate time integration allow to perform computations in groups of different stable time steps with the same ERK method. In this paper, the based method is the second order $2-$stages ERK. Let's consider the example in Fig.1 with two bulk groups of stable time steps Δt_m (red), $2\Delta t_m$ (pink) and separated by a buffer group of stable time step $2\Delta t_m$ (green). The time integration on the multirate partitioning consists in:

1) Apply once the $2-$stages ERK method on bulk group of stable time step $2\Delta t_m$ (pink).
2) Apply twice successively the $2-$stages ERK on the group of $\triangle t_m$. This can be resume in a $2\times 2-$stages ERK method whose butcher tableau is defined by Table.I.
3) For the buffer group of stable time step $2\Delta t_m$ (green), since the ERK method on the bulk group of time step $\triangle t_m$ has $2\times 2-$stages, the based method is adapted to a $2\times 2-$stages ERK method whose Butcher tableau is given by Table.II.

$$
\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
1/2 & 1/4 & 1/4 & & \\
1 & 1/4 & 1/4 & 1/2 & \\
\hline
 & 1/4 & 1/4 & 1/4 & 1/4
\end{array}
$$

TABLE I

BUTCHER TABLEAU OF THE $2 \times 2-$STAGES ERK METHOD ISSUED FROM APPLYING TWICE THE $2-$STAGES ERK.

$$
\begin{array}{c|cccc}
0 & & & & \\
1 & 1 & & & \\
0 & 0 & 0 & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/4 & 1/4 & 1/4 & 1/4
\end{array}
$$

TABLE II

BUTCHER TABLEAU OF THE $2-$STAGES ERK IN FORM OF A $2 \times 2-$STAGES .

IV. NUMERICAL EXAMPLE

We consider a human skull geometry presented in Fig.2. The properties inside the skull are: $\epsilon = 43\epsilon_0$, $\mu = \mu_0$, $\sigma = 1.15S/m$ and $\rho = 1050Kg/m^3$, where ϵ_0 and μ_0 are respectively the permittivity and the permeability of the vacuum. The incident fields $\overrightarrow{E} = (0, 0, E)$ propagates from the square surface located on the left Fig.2. The excitation is a gaussian modulated pulse of center frequency $f_0 = 1.3Ghz$. The simulation time is $T_f = 7.10^{-8}s$. A tetrahedral mesh of 31800 elements is used. The multirate partitioning is formed by ten groups whose half are buffer groups. Simulations are performed on a 8cores$-15Go/$Ram$-2.3Ghz$ computer. The gain obtained with the multirate strategy is presented in TableIII. The specific absorption rate in the volume defined by $SAR_{vol} = \int_{vol} \frac{\sigma \|\overrightarrow{E}\|^2}{\rho} dV$ is computed to show similarity of results obtained with the singlerate and the multirate schemes Fig.3. More details on the method and results will be provided in an extended version of this paper.

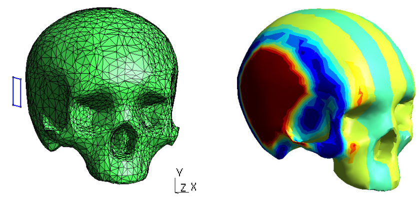

Fig. 2. (left) Human skull geometry (right) The electric field that propagates through the skull

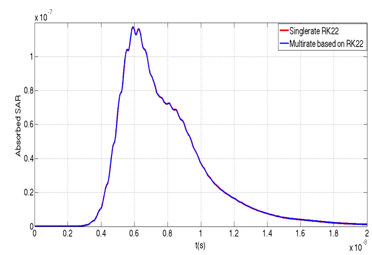

Fig. 3. Comparison of the specific absorption rate in the volume obtained using the singlerate and the multirate schemes with first order spatial mesh elements.

	Singlerate		**Multirate**		
spatial order	Δt	CPU	Δt	CPU	Speedup
1	$2.3e-14$	$7.2e5$	$7.4e-13$	$1e5$	7.2
2	$1.2e-14$	$3.1e6$	$3.9e-13$	$4.1e5$	7.6
3	$7.7e-15$	$9.3e6$	$2.4e-13$	$1.2e6$	7.5

TABLE III

COMPARISON OF THE SINGLERATE AND MULTIRATE PERFORMANCES.

REFERENCES

[1] Schomann, S.; Warburton, T.; Clemens, M.; *Local Timestepping Techniques Using Taylor Expansion for Modeling Electromagnetic Wave Propagation With Discontinuous Galerkin-FEM* , IEEE Trans. Magn, vol. 46, num. 8, pp.3504-3507, 2010.

[2] S. Descombes, S.; Lanteri, S.; Moya, L.; *Locally Implicit Time Integration Strategies in a Discontinuous Galerkin Method for Maxwell?s Equations*, J. Sc. Com., vol 56, num. 1, pp.190-218, 2013

[3] Constantinescu, E.M.; Sandu, A.; *Multirate time stepping methods for hyperbolic conservation laws*, Journal of Scientific Computing, Vol.33, pp. 239-278, 2007.

[4] Godel, N.; Schomann, S.; Warburton, T.; Clemens, M.; *GPU Accelerated Adams-Bashforth Multirate Discontinuous Galerkin FEM Simulation of High-Frequency Electromagnetic Fields* , IEEE Trans. Magn, vol. 46, num. 8, pp.2735-2738, 2010.

[5] Seny, B. and al., *Multirate time stepping for accelerating explicit discontinuous Galerkin computations with application to geophysical flows*, Inter. J. Num. Meth in fluids, vol. 71, pp.41-64, 2013.

[6] Hesthaven, J.S.; Warburton, T.; *Nodal High-Order methods Methods on Unstructured Grids: Time domain solutions of Maxwell's equations*, J. Comp Phys, vol.181, num.1, pp. 186-221, 2002.